Solution 4

1. Prove Hólder's Inequality in vector form: For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $p > 1$ and q conjugate to p,

$$|\mathbf{a} \cdot \mathbf{b}| \leq \left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |b_j|^q \right)^{1/q} .$$

Solution You may imitate the proof for integrable functions in notes. Another way is to deduce it from what we have proved. Let $\mathbf{a}, \mathbf{b}$ be given. Define two integrable functions on $[0,1]$ by setting $f(x) = a_j, x \in [(j-1)/n, j/n], j = 1, \cdots, n$ and similarly $g(x) = b_j, x \in [(j-1)/n, j/n]$. Then $\|fg\|_1 \leq \|f\|_p \|g\|_q$ will turn into $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\|_p \|\mathbf{b}\|_q$.

2. A quick proof of Hölder's Inequality consists of two steps: First, assuming $\|f\|_p = \|g\|_p = 1$ and integrate Young's Inequality. Next, observe that $f/\|f\|_p$ satisfies the first step. Can you find any disadvantage of this approach?

Solution. Following the hint, first assume $\|f\|_p = \|g\|_p = 1$. We apply Young's Inequality to get

$$|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q} .$$

Then integrate, using $\|f\|_p = \|g\|_p = 1$ and $1/p + 1/q = 1$ to get the Hölder's Inequality in the form

$$\int_a^b |fg| \, dx \leq 1.$$

Next, since $F = f/\|f\|_p$ and $G = g/\|g\|_q$ satisfy the conditions in the first step. We have

$$\int_a^b |FG| \, dx \leq 1 .$$

Writing back in f and g, we get the desired Hölder's Inequality.

Note. A disadvantage of this proof, in my opinion, is that it cannot yield the characterization of the case of equality.

3. Prove the generalized Hölder Inequality: For $f_1, f_2, \cdots, f_n \in R[a,b]$,

$$\int_a^b |f_1 f_2 \cdots f_n| dx \leq \left(\int_a^b |f_1|^{p_1} \right)^{1/p_1} \left(\int_a^b |f_2|^{p_2} \right)^{1/p_2} \cdots \left(\int_a^b |f_n|^{p_n} \right)^{1/p_n} ,$$

where

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1, \quad p_1, p_2, \cdots, p_n > 1 .$$

Solution. Induction on n. $n = 2$ is the original Hölder, so it holds. Let

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{n+1}} = 1 .$$

First, using the original Hölder, we have

$$\int_a^b |f_1 f_2 \cdots f_{n+1}| \, dx \leq \left(\int_a^b |f_1|^{p_1} \, dx \right)^{1/p_1} \left(\int_a^b |f_2 \cdots f_{n+1}|^q \, dx \right)^{1/q} ,$$

where q is conjugate to p_1. It is easy to see

$$1 = \frac{q}{p_2} + \cdots + \frac{q}{p_{n+1}} \ .$$

By induction hypothesis,

$$\int_a^b |f_2^q \cdots f_n^q| \, dx \le \left(\int_a^b |f_2|^{p_2} \, dx \right)^{1/p_2} \cdots \left(\int_a^b |f_{n+1}|^{p_{n+1}} \, dx \right)^{1/p_{n+1}} ,$$

done.

4. Establish the inequality, for $f \in R[a,b]$,

$$\int_a^b |f| \, dx \le (b-a)^{1/q} \left(\int_a^b |f|^p \, dx \right)^{1/p} , \quad 1/p + 1/q = 1, p > 1 \ .$$

Solution. This is a special case of the next problem.

5. Establish the inequality, for $f \in R[a,b]$, $\|f\|_{p_1} \le C\|f\|_{p_2}$ when $1 \le p_1 < p_2$.

Solution By Holder's Inequality,

$$\int_a^b |f|^{p_1} \le \left(\int_a^b 1 \, dx \right)^{1-p_1/p_2} \left(\int_a^b |f|^{p_1 \frac{p_2}{p_1}} \, dx \right)^{p_1/p_2} \le C \|f\|_{p_2}^{p_1} ,$$

where

$$C = (b-a)^{\frac{p_2 - p_1}{p_1 p_2}} \ .$$

6. Show that there is no constant C such that $\|f\|_2 \le C\|f\|_1$ for all $f \in C[0,1]$.

Solution Consider again the sequence

$$f_n(x) = \begin{cases} -n^3 x + n, & x \in [0, 1/n^2], \\ 0, & x \in (1/n^2, 1]. \end{cases}$$

We have $\|f_n\|_1 = 1/(2n) \to 0$ as $n \to \infty$, but $\|f_n\|_2 = 1/3$ for all n. Hence, it is impossible to have some C satisfying $\|f\|_2 \le C\|f\|_1$ for all f.

Remark In general, it is impossible to find a constant C such that $\|f\|_{p_2} \le C\|f\|_{p_1}, p_1 < p_2$, for all f.

7. Show that $\|a\| = \left(\sum_j |a_j|^p \right)^{1/p}$ is no longer a norm for $p \in (0,1)$ in $\mathbb{R}^n$.

Solution. Although the first two axioms of a norm hold but the last one is bad. For example, take $a = (1,0), b = (0,1)$ in $\mathbb{R}^2$. We have $|a|_p = |b|_p = 1$ so $|a|_p + |b|_p = 2$ but $|a+b|_p = |(1,1)|_p = 2^{1/p} > |a|_p + |b|_p$, the inequality is reversed !

8. Show that $\|f\|_p$ is no longer a norm on $C[0,1]$ for $p \in (0,1)$.

Solution Again (M3) is bad. Consider two functions $f = \chi_{[0,1/2]}$ and $g = \chi_{[1/2,1]}$. We have $\|f+g\|_p = 1$ but $\|f\|_p = \|g\|_p = 2^{-1/p}$, so $\|f+g\|_p > \|f\|_p + \|g\|_p$. Although f and g are not continuous, we could find continuous approximations to these functions with the same effect.

9. Optional. Show that any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$ are equivalent, that is, there exists two constants C_1, C_2 such that $\|x\|_1 \le C_1\|x\|_1$ and $\|x\|_2 \le C_2\|x\|_1$ for all $x \in \mathbb{R}^n$. Hint: It suffices to show every norm is equivalent to the Euclidean norm.

   Solution It suffices to show that any norm on $\mathbb{R}^n$ is equivalent to the Euclidean norm. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. For $x = \sum \alpha_j e_j$, recalling that $\|x\|_2 = \sqrt{\sum |\alpha_j|^2}$, we have

$$\|x\| \le \sum |\alpha_j|\|e_j\| \le \sqrt{\sum |\alpha_j|^2}\sqrt{\sum \|e_j\|^2} = C\|x\|_2.$$

   This shows that $\|\cdot\|_2$ is stronger than $\|\cdot\|$. To establish the other inequality, letting $\varphi(x) = \|x\|$, from the triangle inequality $|\varphi(x) - \varphi(y)| \le \|x-y\| \le C\|x-y\|_2$, φ is a continuous function. Consider

$$\alpha \equiv \inf\{\varphi(x) : x \in \mathbb{R}^n, \|x\|_2 = 1\}.$$

   As the function φ is positive on the unit sphere of $\|\cdot\|_2$, α is a nonnegative number. The second inequality will come out easily if α is positive. To see this we observe that for every nonzero $x \in \mathbb{R}^n$,

$$0 < \alpha \le \varphi\left(\frac{x}{\|x\|_2}\right) = \frac{\|x\|}{\|x\|_2},$$

   i.e.,

$$\alpha\|x\|_2 \le \|x\|, \ \forall x.$$

   To show that α is positive, we use the fact that every continuous function on a closed and bounded subset of $\mathbb{R}^n$ must attain its minimum. Applying it to φ and the unit sphere $\{\|x\|_2 = 1\}$, the infimum α is attained at some point x_0 and so in particular $\alpha = \varphi(x_0) > 0$.

10. Let l^p consist of all sequences $\{a_n\}$ satisfying $\sum_n |a_n|^p < \infty$. Show that

$$\|a\|_p = \left(\sum_n |a_n|^p\right)^{1/p},$$

   defines a norm on $l^p, 1 \le p < \infty$. Propose a definition for the metric space l^∞.

   Fix n. By the Minkowski inequality in $\mathbb{R}^n$,

$$\begin{aligned}\left(\sum_{j=1}^n |a_j + b_j|^p\right)^{1/p} &\le \left(\sum_{j=1}^n |a_j|^p\right)^{1/p} + \left(\sum_{j=1}^n |b_j|^p\right)^{1/p} \\ &\le \left(\sum_{j=1}^\infty |a_j|^p\right)^{1/p} + \left(\sum_{j=1}^\infty |b_j|^p\right)^{1/p},\end{aligned}$$

   and the desired triangle inequality in l^p follows by letting $n \to \infty$.

   One can define l^∞ to be the vector space consisting of all bounded sequences. It is a normed one under $\|a\| = \sup_{j\ge1} |a_j|$.

11. Define d on $\mathbb{Z}\times\mathbb{Z}$ by $d(n,m) = 2^{-d}$, where d is the largest power of 2 dividing $n - m \ne 0$ and set $d(n,n) = 0$. Verify that d defines a metric on $\mathbb{Z}$.

   Solution. Noticing that the function d is positive unless $n = m$, $(M3)$ and $(M2)$ are clearly satisfied. If 2^d divides $m-k$ and $k-n$, then 2^d divides $m - n = m - k + k - n$. Hence

$$d(m,n) \le \max(d(m,k), d(k,n)) \le d(m,k) + d(k,n),$$

   and $(M3)$ is also satisfied.